# Section 15.1
## Integration in Two Variables





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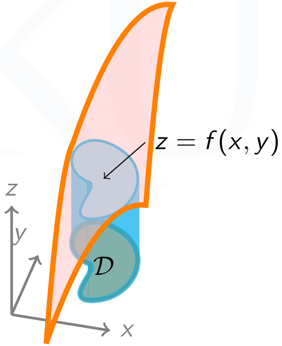
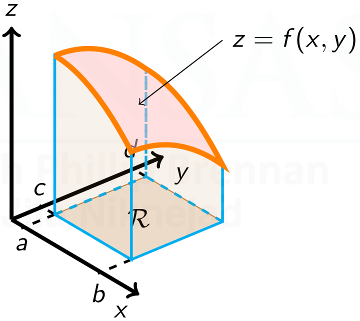
# 1 Volume Under the Graph of a 2-Variable Function

# Volume Under $f(x, y)$

Let $f(x, y)$ be a function of two variables on a bounded domain $\mathcal{D}$ in $\mathbb{R}^2$.

**Problem: Find the volume of the solid $W$ between the graph of $f$ and the domain $\mathcal{D}$.**




We will start with the case of a rectangular domain $\mathcal{R} = [a, b] \times [c, d]$.

# Integration in Calculus I and Calculus III

Link

| | **Calculus I** | **Calculus III** |
|---|---|---|
| **Function** | $y = f(x)$<br>on $[a, b]$ | $z = f(x, y)$<br>on $D$ |
| **Problem** | Find area under curve | Find volume under surface |
| **Idea** | Slice into thin rectangles<br>of height $f(x)$<br>and width $\Delta x$ | Slice into thin prisms<br>of height $f(x, y)$<br>and base $\Delta x \, \Delta y$ |
| **Approximation** | Riemann sum<br>$\displaystyle\sum_{i=1}^{N} f(x_i^*) \, \Delta x$ | Riemann sum<br>$\displaystyle\sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i^*, y_j^*) \, \Delta x \, \Delta y$ |
| **Final answer** | Single integral<br>$\displaystyle\int_a^b f(x) \, dx$ | Double integral<br>$\displaystyle\iint_D f(x, y) \, dA$ |

# Double Integrals Over a Rectangle, a Riemann Sum $S_{M,N}$

Let $f(x, y)$ be a function on the rectangle $\mathcal{R} = [a, b] \times [c, d]$.
For now, assume $f(x, y) \geq 0$ on $R$.

(1) Partition $R$ into small rectangles $R_{ij}$ by dividing $[a, b]$ into $N$ equal subintervals and $[c, d]$ into $M$ equal subintervals:

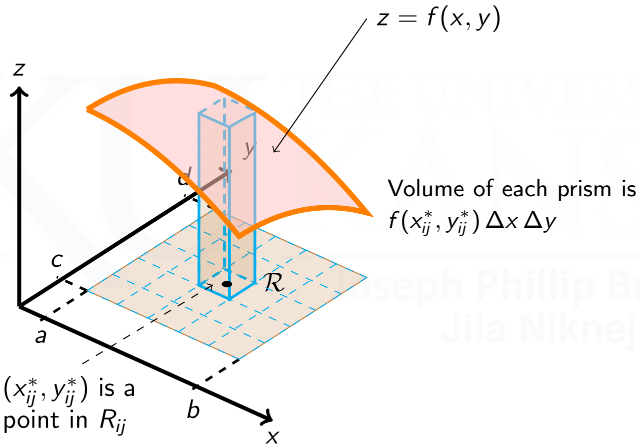
$$\begin{aligned} a &= x_0 < x_1 < \ldots < x_{N-1} < x_N = b \\ c &= y_0 < y_1 < \ldots < y_{M-1} < y_M = d \end{aligned} \qquad R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

Each rectangle $R_{ij}$ has sides of lengths $\Delta x = \frac{b-a}{N}$ and $\Delta y = \frac{d-c}{M}$.

(2) Choose a point $P_{ij} = (x_{ij}^*, y_{ij}^*)$ in the rectangle $R_{ij}$. Let $B_{ij}$ be the rectangular prism with base $R_{ij}$ and height $f(P_{ij})$.

$$\begin{aligned} \text{Volume}(B_{ij}) &= f(x_{ij}^*, y_{ij}^*)\,\Delta x\,\Delta y = f(x_{ij}^*, y_{ij}^*)\,\Delta A \\ &\approx \ \text{volume under the graph of } f(x, y) \text{ and over } R_{ij} \end{aligned}$$

# Double Integrals



# Double Integrals

(3) The volume of $W$ **is approximately** the sum of the volumes of the boxes $B_{ij}$ (a double Riemann sum):

$$Volume(W) \approx \sum_{i=1}^{N} \sum_{j=1}^{M} Volume(B_{ij}) = \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{M} f(x_{ij}^*, y_{ij}^*)\, \Delta A}_{S_{M,N}}$$

(4) The volume of $W$ **equals** the limit as both $N$ and $M$ approach $\infty$.

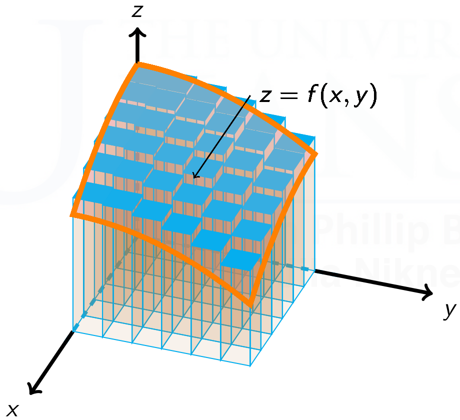
$$Volume(W) = \lim_{N\to\infty} \lim_{M\to\infty} \sum_{i=1}^{N} \sum_{j=1}^{M} Volume(B_{ij}) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

If the limit exists, we call it the **double integral** of $f$ over $R$, denoted by

$$\iint_R f(x, y)\, dA.$$

# Double Integrals

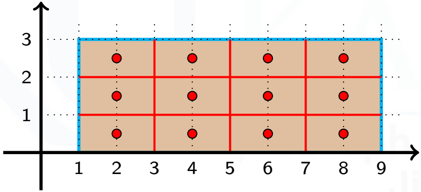
$$\iint_R f(x,y)\,dA = \lim_{N\to\infty}\lim_{M\to\infty}\sum_{i=1}^{N}\sum_{j=1}^{M} Volume(B_{ij})$$



# Riemann Sums — Example

**Example 1:** Approximate $\iint_{[1,9]\times[0,3]} xy\,dA$ as a Riemann sum $S_{4,3}$, using the midpoint of each rectangle as the sample point. Repeat with the upper left points.

<u>Solution:</u> Draw the domain of integration and split it up into a $4 \times 3$ grid of rectangles.




$\Delta x = 2$

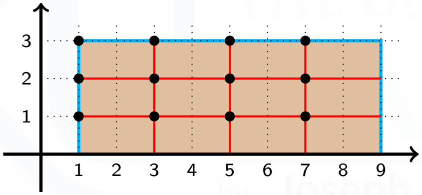
$\Delta y = 1$

$\Delta A = 2$

$$\begin{aligned} S_{4,3} = \Big(&2 \times 0.5 + 4 \times 0.5 + 6 \times 0.5 + 8 \times 0.5 \\ +\ &2 \times 1.5 + 4 \times 1.5 + 6 \times 1.5 + 8 \times 1.5 \\ +\ &2 \times 2.5 + 4 \times 2.5 + 6 \times 2.5 + 8 \times 2.5\Big)\Delta A = 180. \end{aligned}$$

# Riemann Sums, an Example

**Example 1 (cont'd):** Approximate $\iint_{[1,9]\times[0,3]} xy\, dA$ as a Riemann sum $S_{4,3}$ using the upper left end points. Solution: Draw the domain of integration and split it up into a $4 \times 3$ grid of rectangles.




$$\Delta x = 2$$
$$\Delta y = 1$$
$$\Delta A = 2$$

$$S_{4,3} = \Big(1 \times 1 + 3 \times 1 + 5 \times 1 + 7 \times 1 \\ + \; 1 \times 2 + 3 \times 2 + 5 \times 2 + 7 \times 2 \\ + \; 1 \times 3 + 3 \times 3 + 5 \times 3 + 7 \times 3\Big)\Delta A = 192.$$

# 2 Using Iterated Integrals to Compute

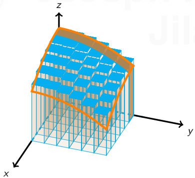
# Computing Double Integrals

The double integral is defined the same way for functions $f(x, y)$ that are not necessarily positive on the domain $R$:

$$\iint_R f(x, y)\,dA = \text{volume above } R \text{ and below graph of } f$$
$$- \text{volume below } R \text{ and above graph of } f$$

Definite integrals can be defined over general regions, not just above rectangles. We'll see this in Section 15.2.

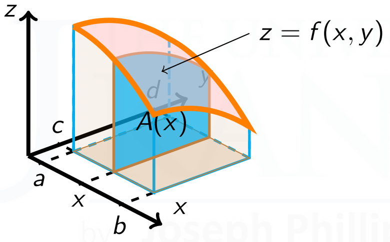
**Question:** How do we *calculate* double integrals in practice? Link



# Iterated Integrals: The Intuition

Assume $f(x, y) \geq 0$. Let $W$ be the solid region below the graph of $f(x, y)$ and above the rectangle $R = [a, b] \times [c, d]$.

We can cut $W$ into "slices" $W_x$ for each $x \in [a, b]$.




For a fixed $x$, the slice $W_x$ has area $A(x) = \int_c^d f(x, y)\,dy$.

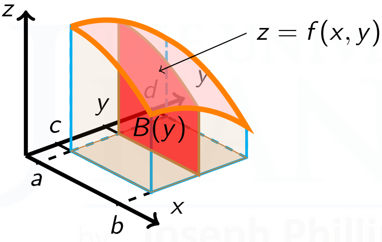
The volume of $W$ is the integral of the cross-sectional areas $W_x$:

$$Volume(W) = \iint_R f(x, y)\,dA = \int_a^b A(x)\,dx = \int_a^b \left( \int_c^d f(x, y)\,dy \right) dx$$

# Iterated Integrals: The Intuition

Assume $f(x, y) \geq 0$. Let $W$ be the solid region below the graph of $f(x, y)$ and above the rectangle $R = [a, b] \times [c, d]$.

We can cut $W$ into "slices" $W_y$ for each $y \in [c, d]$.






For a fixed $y$, the slice $W_y$ has area $B(y) = \int_a^b f(x, y)\, dx$.

The volume of $W$ is the integral of the cross-sectional areas $W_y$:

$$Volume(W) = \iint_R f(x, y)\, dA = \int_c^d B(y)\, dy = \int_c^d \left( \int_a^b f(x, y)\, dx \right) dy$$

Link

# Iterated Integrals: Fubini's Theorem

## Fubini's Theorem

If $f(x, y)$ is continuous on a rectangle $R = [a, b] \times [c, d]$ then

$$\iint_R f(x, y)\, dA = \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} f(x, y)\, dy \right) dx$$
$$= \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x, y)\, dx \right) dy$$

The integrals on the right-hand-side are called **iterated integrals**.

To evaluate the iterated integral $\int_a^b \int_c^d f(x, y)\, dy\, dx$:

(i) Evaluate the inner integral for $y$ by treating $x$ as a fixed parameter.

The result of $\int_c^d f(x, y)\, dy$ is a **function** of $x$.

(ii) Then evaluate the integral with respect to $x$.

# Fubini's Theorem

**Example 2a:** Let $R = [0,1] \times [0,1]$. Evaluate $\iint_R (1 + x + 2y)\, dA$.

Solution: The function is continuous, so by Fubini's Theorem



$$\iint_R (1 + x + 2y)\, dA = \int_0^1 \left( \int_0^1 (1 + x + 2y)\, dy \right) dx$$

$$= \int_0^1 \left( y + xy + y^2 \right) \Big|_{y=0}^{y=1} dx = \int_0^1 (x + 2)\, dx$$

$$= \left( 2x + \frac{x^2}{2} \right) \Big|_{x=0}^{x=1} = 5/2.$$

# Fubini's Theorem

**Example 2b:** Let $R = [0,1] \times [0,1]$. Evaluate $\displaystyle\iint_R (1 + x + 2y)\, dA$.

<u>Solution:</u> The function is continuous, so by Fubini's Theorem

$$
\begin{aligned}
\iint_R (1 + x + 2y)\, dA &= \int_0^1 \left( \int_0^1 (1 + x + 2y)\, dx \right) dy \\
= \int_0^1 \left( x + \frac{x^2}{2} + 2xy \right) \Bigg|_{x=0}^{x=1} dy &= \int_0^1 \left( \frac{3}{2} + 2y \right) dy \\
= \left( \frac{3y}{2} + y^2 \right) \Bigg|_{y=0}^{y=1} &= 2(1) + \frac{1^2}{2} = 5/2.
\end{aligned}
$$

# Split Double Integrals

**Example 3:** Evaluate $\displaystyle\iint_{[0,2]\times[1,3]} x^2 y\,dA$.

Solution: For this integral, both orders of iteration work the same way:

$$\int_0^2 \int_1^3 x^2 y\,dy\,dx = \int_0^2 x^2 \left(\int_1^3 y\,dy\right) dx \quad \textbf{or} \quad \int_1^3 y \left(\int_0^2 x^2\,dx\right) dy$$

$$= \left(\int_0^2 x^2\,dx\right)\left(\int_1^3 y\,dy\right)$$

$$= \left(\frac{x^3}{3}\right)\Bigg|_0^2 \left(\frac{y^2}{2}\right)\Bigg|_1^3 = \left(\frac{8}{3}\right)\left(\frac{1}{2}\right)(9-1) = \frac{32}{3}.$$

**Split Double Integrals**

$$\iint_R f(x)g(y)\,dA = \left(\int_b^a f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right).$$

(Careful! This rule **only** applies if the integrand has the form $f(x)g(y)$.)

Proof: $\displaystyle\iint_R f(x)g(y)\,dA = \int_c^d \int_a^b f(x) \underbrace{g(y)}_{\text{constant}}\,dx\,dy = \int_c^d g(y) \underbrace{\int_a^b f(x)\,dx}_{\text{constant}}\,dy = \left(\int_b^a f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$

# Choosing the Order of Iteration

**Example 4:** Evaluate $\displaystyle\iint_{[0,2]\times[1,2]} xe^{xy}\,dA$.

Solution: Integrating first with respect to $x$ would require a messy integration by parts. It is easier to integrate first with respect to $y$:

$$\iint_{[0,2]\times[1,2]} xe^{xy}\,dA = \int_0^2 \left(\int_1^2 xe^{xy}\,dy\right)dx$$

$$= \int_0^2 \left(e^{xy}\right)\Big|_{y=1}^{y=2} dx = \int_0^2 (e^{2x} - e^{x})\,dx$$

$$= \left(\frac{e^{2x}}{2} - e^{x}\right)\Big|_{x=0}^{x=2} = \left(\frac{e^{2(2)}}{2} - e^{2}\right) - \underbrace{\left(\frac{e^{0}}{2} - e^{0}\right)}_{=1/2} = \frac{e^{4}}{2} - e^{2} + \frac{1}{2}$$